Modeling and simulation of multilane traffic flow

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A most important aspect in the field of traffic modeling is the simulation of bottleneck situations. For their realistic description a macroscopic multilane model for unidirectional freeways including acceleration, deceleration, velocity fluctuations, overtaking, and lane-changing maneuvers is systematically deduced from a gas-kinetic (Boltzmann-like) approach. The resulting equations contain corrections with respect to previous models. For efficient computer simulations, a reduced model delineating the coarse-grained temporal behavior is derived and applied to bottleneck situations. [S1063-651X(97)01305-6]

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I. INTRODUCTION

Apart from microscopic traffic models, in the last four decades a number of interrelated macroscopic traffic models have been proposed [1–10]. The motivations for developing these were to describe and understand the instabilities of traffic flow [5–8,10–12]; to optimize traffic flow by means of on-line speed-control systems [13–15]; to make short-term forecasts of traffic volumes for rerouting measures [16–18]; to calculate the average travel times, fuel consumption, and vehicle emissions in dependence on traffic volume [4,19]; and to predict the effects of additional roads or lanes [19–22]. Most of these models are restricted to unidirectional freeway traffic and treat the different lanes of a road in an overall manner, i.e., like one lane with higher capacity and possibilities for overtaking. However, this kind of simplification is clearly not applicable if there is a disequilibrium between neighboring lanes. Therefore some researchers carried out empirical investigations of the observed density oscillations between neighboring lanes or proposed models for their mutual influences [23–28].

However, these are phenomenological models which treat interlane interactions in a rather heuristic way. Moreover, most of them base on the simple traffic flow model of Lighthill and Whitham which assumes average velocity on each lane to be in equilibrium with density. This assumption is not very well justified, especially for unstable traffic which is characterized by evolving stop-and-go waves [5–8,10–12]. It is also questionable for lane mergings or on-ramp traffic where frequently a disequilibrium occurs [5,8]. However, instabilities or disequilibria may decrease the freeway capacity considerably.

Another approach including a phenomenological velocity equation has been proposed by Michalopoulos *et al.* [28]. It bases on Payne's model [3,4], which has been severely criticized for several reasons [5,10,29–34]. Therefore we will derive a consistent macroscopic multilane model from a *gas-kinetic* level of description. This is related to Paveri-Fontana's approach (cf. Sec. II), but explicitly takes into account overtaking and lane-changing maneuvers. The corresponding Boltzmann-like model allows one to deduce macroscopic traffic equations not only for the vehicle densities on the different lanes, but also for the associated average velocities (cf. Sec. III). Due to different legal regulations, the traffic dynamics on American freeways is different from that on European ones (which will be called *''autobahns''* in accordance with Kühne and Rödiger [13,6]).

For efficient computer simulations of large parts of a freeway system it is desirable to have a somewhat simpler model. Therefore in Sec. IV we will eliminate the velocity equations and derive a reduced multilane model for the traffic dynamics on a slow time scale. By means of computational results it is demonstrated that even the difficult bottleneck situations can be successfully simulated with this model.

A summary and outlook is presented in Sec. V. However, before the *macroscopic* multilane model sketched in Ref. [35] is founded, derived, simplified, and simulated, the alternative *microscopic* approaches shall be mentioned. One class of microsimulation models bases on cellular automata, like the ones by Rickert *et al.* [36] and Nagatani [37]. These update the vehicle dynamics within two successive steps: either the vehicle motion in one step and lane changing in the next step [36], or the left lane in the first step and the right lane in the second one [37]. Bottlenecks were represented by crashed cars with zero velocity [37]. Noteworthy are also the event-oriented model by Wiedemann and Benz [38] and the social force model by Helbing and Schwarz [19].

II. BOLTZMANN-LIKE MULTILANE THEORY

The first Boltzmann-like (gas-kinetic) model was proposed by Prigogine and co-workers [39–41]. However, Paveri-Fontana [42] has pointed out that this model has some peculiar properties. For this reason, Paveri-Fontana proposed an improved model that overcomes most of the shortcomings of Prigogine's approach. Nevertheless, his model still treats the lanes of a multilane road in an overall manner. Therefore an extended Paveri-Fontana-like model will now be constructed.

Let us assume that the motion of an individual vehicle α can be described by several variables such as its *lane* $i_\alpha(t)$, its *place* $r_\alpha(t)$, its *actual velocity* $v_\alpha(t)$, and its *desired velocity* $v_{0\alpha}(t)$ in dependence on time t. The *phase-space density* $\hat{\rho}_i(r,v,v_0,t)$ is then determined by the number $\Delta n_i(r,v,v_0,t)$ of vehicles on lane i that are at a place between $r-\Delta r/2$ and $r+\Delta r/2$, driving with a velocity between $v-\Delta v/2$ and $v+\Delta v/2$, and having a desired velocity between $v_0-\Delta v_0/2$ and $v_0+\Delta v_0/2$ at time t:

$$\hat{\rho}_i(r,v,v_0,t)=\frac{\Delta n_i(r,v,v_0,t)}{\Delta r\Delta v\Delta v_0}=\frac{1}{\Delta r\Delta v\Delta v_0}\sum_\alpha \delta_{ii_\alpha(t)} \times\int_{r-\Delta r/2}^{r+\Delta r/2}dr'\,\delta(r'-r_\alpha(t)) \times\int_{v-\Delta v/2}^{v+\Delta v/2}dv'\,\delta(v'-v_\alpha(t)) \times\int_{v_0-\Delta v_0/2}^{v_0+\Delta v_0/2}dv_0'\,\delta(v_0'-v_\alpha^0(t)). \quad (1)$$

Here, Δr, Δv, and Δv_0 are small intervals. δ_{ij} denotes the Kronecker symbol and $\delta(x-y)$ Dirac's delta function. The notation ''t'' indicates that a time dependence only occurs in exceptional cases. Lane numbers i are counted in increasing order from the right most to the left most lane, but in Great Britain and Australia the other way around. (For Great Britain and Australia ''left'' and ''right'' must always be interchanged.)

Now we utilize the fact that, due to the conservation of the number of vehicles, the phase-space density $\hat{\rho}_i(r,v,v_0,t)$ on lane i obeys the continuity equation [19,42,43]

$$\frac{\partial\hat{\rho}_i}{\partial t}+\frac{\partial}{\partial r}(\hat{\rho}_i v)+\frac{\partial}{\partial v}(\hat{\rho}_i f_i^0) =\left(\frac{\partial\hat{\rho}_i}{\partial t}\right)_{\text{ad}}+\left(\frac{\partial\hat{\rho}_i}{\partial t}\right)_{\text{int}}+\left(\frac{\partial\hat{\rho}_i}{\partial t}\right)_{\text{LC}}+\hat{\nu}_i^+(r,v,v_0,t) -\hat{\nu}_i^-(r,v,v_0,t). \quad (2)$$

The second and third terms describe temporal changes of the phase-space density $\hat{\rho}_i(r,v,v_0,t)$ due to changes $dr/dt=v$ of place r and due to acceleration f_i^0, respectively. We will assume that the vehicles accelerate to their desired velocity v_0 with a certain, density-dependent relaxation time τ_i, so that we have the acceleration law

$$f_i^0(r,v,v_0,t)=\frac{v_0-v}{\tau_i}. \quad (3)$$

The terms on the right-hand side of Eq. (2) reflect changes of phase-space density $\hat{\rho}_i(r,v,v_0,t)$ due to *discontinuous* changes of desired velocity v_0, actual velocity v, or lane i. $\hat{\nu}_i^+(r,v,v_0,t)$ and $\hat{\nu}_i^-(r,v,v_0,t)$ are the rates of vehicles entering and leaving the road at place r. They are only different from zero for merging lanes at entrances and exits, respectively.

The term

$$\left(\frac{\partial\hat{\rho}_i}{\partial t}\right)_{\text{ad}}=\frac{\tilde{\rho}_i(r,v,t)}{T_{\text{r}}}[\hat{P}_{0i}(v_0;r,t)-P_{0i}(v_0;r,t)], \quad (4)$$

where

$$\tilde{\rho}_i(r,v,t)=\int dv_0\hat{\rho}_i(r,v,v_0,t) \quad (5)$$

is a reduced phase-space density and $T_r\approx 1$ s is about the reaction time, describes an adaptation of the *actual* distribution of desired velocities $P_{0i}(v_0;r,t)$ to the *reasonable* distribution of desired velocities $\hat{P}_{0i}(v_0;r,t)$ without any related change of actual velocity v.

For the reasonable distribution of desired velocities we will assume the functional dependence

$$\hat{P}_{0i}(v_0;r,t)=\frac{1}{\sqrt{2\pi\hat{\theta}_{0i}}}e^{-(v_0-\hat{V}_{0i})^2/(2\hat{\theta}_{0i})}, \quad (6)$$

which corresponds to a normal distribution and is empirically well justified [44,24,19]. The mean value $\hat{V}_{0i}=\hat{V}_{0i}(r,t)$ and variance $\hat{\theta}_{0i}=\hat{\theta}_{0i}(r,t)$ of $\hat{P}_{0i}(v_0;r,t)$ depend on road conditions and speed limits. Since European autobahns usually do not have speed limits (at least in Germany), $\hat{\theta}_{0i}$ is larger for these than for American freeways. In addition, on European autobahns $\hat{V}_{0i}$ increases with increasing lane number i since overtaking is only allowed in the left-hand lane.

Before we specify the Boltzmann-like interaction term $(\partial\hat{\rho}_i/\partial t)_{\text{int}}$ and the lane-changing term $(\partial\hat{\rho}_i/\partial t)_{\text{LC}}$ we will discuss some preliminaries. For reasons of simplicity we will only treat vehicle interactions within the *same* lane as *direct pair interactions*, i.e., in a Boltzmann-like manner [45]. Lane-changing maneuvers of impeded vehicles that want to escape a queue (i.e., leave and overtake it) may depend on interactions of up to six vehicles (the envisaged vehicle, the vehicle directly in front of it, and up to two vehicles on both neighboring lanes which may prevent overtaking if they are too close). Therefore we will treat lane-changing maneuvers in an overall manner by specifying overtaking probabilities and waiting times of lane-changing maneuvers (which corresponds to a *mean-field approach*, see Ref. [45]). These probabilities and waiting times are dependent on the vehicle densities and may also depend on other quantities.

For not too large vehicle densities the Boltzmann-like interaction term can be written in the form [19]

$$\left(\frac{\partial\hat{\rho}_i}{\partial t}\right)_{\text{int}}=\sum_{i'}\int dv'\int_{w<v'}dw\int dw_0 W_2(v,i|v',i';w,i') \times\hat{\rho}_{i'}(r,v',v_0,t)\hat{\rho}_{i'}(r,w,w_0,t) \quad (7a)$$

$$-\sum_{i'}\int dv'\int_{w<v}dw\int dw_0 W_2(v',i'|v,i;w,i) \times\hat{\rho}_i(r,v,v_0,t)\hat{\rho}_i(r,w,w_0,t). \quad (7b)$$

Term (7a) describes an increase of phase-space density $\hat{\rho}_i(r,v,v_0,t)$ by interactions of a vehicle with actual velocity v' and desired velocity v_0 on line i' with a slower vehicle with actual velocity $w<v'$ and desired velocity w_0 causing the former vehicle to change its velocity to $v<v'$ or its lane to $i\neq i'$. The frequency of such interactions is proportional to the phase-space density $\hat{\rho}_{i'}(r,w,w_0,t)$ of hindering ve-

hicles and the phase-space density $\hat{\rho}_{i'}(r,v',v_0,t)$ of vehicles which can be affected by slower vehicles. Analogously, term (7b) describes a decrease of phase-space density $\hat{\rho}_i(r,v,v_0,t)$ by interactions of a vehicle with actual velocity v and desired velocity v_0 on line i with a slower vehicle with actual velocity $w<v$ and desired velocity w_0 causing the former vehicle to change its velocity to $v'<v$ or its lane to $i'\neq i$. Since the interaction is assumed not to influence the desired velocities v_0, w_0, the interaction rate W_2 is independent of these. However, the interaction rate $W_2(v',i'|v,i;w,i)$ is proportional to the relative velocity $|v-w|$ of approaching vehicles. Therefore we have the following relation:

$$W_2(v',i'|v,i;w,i)=p_i^+|v-w|\delta_{i'(i+1)}\delta(v'-v) \tag{8a}$$

$$+p_i^-|v-w|\delta_{i'(i-1)}\delta(v'-v) \tag{8b}$$

$$+(1-p_i)|v-w|\delta_{i'i}\delta(v'-w). \tag{8c}$$

Term (8a) describes an undelayed overtaking in lane $i'=i+1$ without any change of velocity ($v'=v$) by vehicles which would be hindered by slower vehicles in lane i. p_i^+ denotes the corresponding probability of immediate overtaking. Analogously, term (8b) reflects undelayed overtaking maneuvers in lane $i'=i-1$ with probability p_i^-. Term (8c) with

$$p_i=p_i^+ + p_i^- \tag{9}$$

delineates situations where a vehicle cannot be immediately overtaken by a faster vehicle so that the latter must stay in the same lane ($i'=i$) and decelerate to the velocity $v'=w$ of the hindering vehicle.

We come now to the specification of the lane-changing term $(\partial\hat{\rho}_i/\partial t)_{\mathrm{LC}}$. This has the form of a master equation:

$$\left(\frac{\partial\hat{\rho}_i}{\partial t}\right)_{\mathrm{LC}}=\sum_{i'(\neq i)} W_1(i|i')\hat{\rho}_{i'}(r,v,v_0,t) \tag{10a}$$

$$-\sum_{i'(\neq i)} W_1(i'|i)\hat{\rho}_i(r,v,v_0,t). \tag{10b}$$

Term (10a) describes an increase of phase-space density $\hat{\rho}_i(r,v,v_0,t)$ due to changes from lane $i'\neq i$ to lane i by vehicles with actual velocity v and desired velocity v_0. The frequency of lane changing maneuvers is proportional to the phase-space density $\hat{\rho}_{i'}(r,v,v_0,t)$ of vehicles which may be interested in lane-changing. Analogously, term (10b) reflects changes from lane i to another lane i' causing a decrease of $\hat{\rho}_i(r,v,v_0,t)$. For the corresponding rate $W_1(i'|i)$ of lane-changing maneuvers we have the relation

$$W_1(i'|i)=\frac{1}{T_i^+}\delta_{i'(i+1)}+\frac{1}{T_i^-}\delta_{i'(i-1)}, \tag{11}$$

since vehicles can only change to the neighboring lanes $i'=i\pm 1$. T_i^+ (T_i^-) denotes the *waiting times* for *delayed* overtaking in or spontaneous lane-changing maneuvers to the left-hand (right-hand) lane.

Due to different legal regulations, the explicit form of the overtaking probabilities $p_i^\pm$ and the waiting times $T_i^\pm$ in dependence on the vehicle densities is different in American countries than in European ones. A more detailed discussion of this aspect is presented in Ref. [19].

III. DERIVATION OF MACROSCOPIC TRAFFIC EQUATIONS

The gas-kinetic traffic equations are not very suitable for computer simulations since they contain too many variables. Moreover, the phase-space densities are very small quantities and therefore subject to considerable fluctuations so that a comparison with empirical data is difficult. However, the special value of gas-kinetic traffic equations is that they allow a systematic derivation of dynamic equations for the macroscopic (collective) quantities one is mainly interested in.

A. Definition of variables

The most relevant macroscopic quantities are the vehicle densities

$$\rho_i(r,t)=\int dv \int dv_0 \hat{\rho}_i(r,v,v_0,t) \tag{12}$$

and the average velocities

$$V_i(r,t)\equiv\langle v\rangle_i=\int dv\, v P_i(v;r,t) \tag{13}$$

in lanes i. Here, we have applied the notation

$$F_i(r,t)\equiv\langle f(v,v_0)\rangle_i=\int dv\int dv_0 f(v,v_0)\frac{\hat{\rho}_i(r,v,v_0,t)}{\rho_i(r,t)} \tag{14}$$

and introduced the distribution of actual velocities

$$P_i(v;r,t)=\int dv_0\frac{\hat{\rho}_i(r,v,v_0,t)}{\rho_i(r,t)}=\frac{\tilde{\rho}_i(r,v,t)}{\rho_i(r,t)} \tag{15}$$

in lane i. Analogous quantities can be defined for vehicles entering and leaving the road at entrances and exits, respectively.

$$\nu_i^\pm(r,t)=\int dv\int dv_0 \nu_i^\pm(r,v,v_0,t) \tag{16}$$

are the rates of entering and leaving vehicles, and

$$V_i^\pm(r,t)\equiv\langle v\rangle_i^\pm=\int dv\, v P_i^\pm(v;r,t) \tag{17}$$

their average velocities, where

$$P_i^\pm(v;r,t)=\int dv_0\frac{\hat{\nu}_i^\pm(r,v,v_0,t)}{\nu_i^\pm(r,t)} \tag{18}$$

are the velocity distributions of entering and leaving vehicles, respectively. In addition, we will need the velocity variance

$$\theta_i(r,t) \equiv \langle [v - V_i(r,t)]^2 \rangle_i = \int dv [v - V_i(r,t)]^2 P_i(v;r,t) = \langle v^2 \rangle_i - (\langle v \rangle_i)^2 \quad (19)$$

and the average desired velocity

$$V_{0i}(r,t) = \int dv \int dv_0 v_0 \frac{\hat{\rho}_i(r,v,v_0,t)}{\rho_i(r,t)} \quad (20)$$

on each lane i as well as the average interaction rate

$$\frac{1}{T_i^0} = \frac{1}{\rho_i(r,t)} \int dv \tilde{\rho}_i(r,v,t) \int_{w<v} dw (v-w) \tilde{\rho}_i(r,w,t) \quad (21)$$

of a vehicle in lane i with other vehicles in the same lane.

B. Derivation of moment equations

We are now ready to derive the desired macroscopic traffic equations from the gas-kinetic equation (2) with Eqs. (3), (4), (7), (8), (10), and (11). Integration with respect to v_0 gives us the reduced gas-kinetic traffic equation

$$\frac{\partial \tilde{\rho}_i}{\partial t} + \frac{\partial}{\partial r}(\tilde{\rho}_i v) + \frac{\partial}{\partial v}\left(\tilde{\rho}_i \frac{\tilde{V}_{0i}(v) - v}{\tau_i} \right) \quad (22a)$$

$$= -(1-p_i)\tilde{\rho}_i(r,v,t) \int dw (v-w) \tilde{\rho}_i(r,w,t) \quad (22b)$$

$$+ p_{i-1}^+ \tilde{\rho}_{i-1}(r,v,t) \int_{w<v} dw (v-w) \tilde{\rho}_{i-1}(r,w,t) \quad (22c)$$

$$+ p_{i+1}^- \tilde{\rho}_{i+1}(r,v,t) \int_{w<v} dw (v-w) \tilde{\rho}_{i+1}(r,w,t) \quad (22d)$$

$$-(p_i^+ + p_i^-) \tilde{\rho}_i(r,v,t) \int_{w<v} dw (v-w) \tilde{\rho}_i(r,w,t) \quad (22e)$$

$$+ \frac{1}{T_{i-1}^+} \tilde{\rho}_{i-1}(r,v,t) - \frac{1}{T_i^+} \tilde{\rho}_i(r,v,t) \quad (22f)$$

$$+ \frac{1}{T_{i+1}^-} \tilde{\rho}_{i+1}(r,v,t) - \frac{1}{T_i^-} \tilde{\rho}_i(r,v,t) \quad (22g)$$

$$+ \tilde{\nu}_i^+(r,v,t) - \tilde{\nu}_i^-(r,v,t), \quad (22h)$$

with

$$\tilde{V}_{0i}(v) \equiv \tilde{V}_{0i}(v;r,t) = \int dv_0 v_0 \frac{\hat{\rho}_i(r,v,v_0,t)}{\tilde{\rho}_i(r,v,t)} \quad (23)$$

and

$$\tilde{\nu}_i^{\pm}(r,v,t) = \int dv_0 \hat{\nu}_i^{\pm}(r,v,v_0,t). \quad (24)$$

In formula (22), the deceleration term (22b) stems from Eq. (8c), the terms (22c)–(22e) reflecting immediate overtaking come from Eqs. (8a) and (8b), and the lane-changing terms (22f), (22g) originate from Eq. (11). The adaptation term $(\partial \hat{\rho}_i / \partial t)_{\text{ad}}$ yields no contribution.

We will now derive equations for the moments $\langle v^k \rangle$ by multiplying Eq. (22) with v^k and integrating with respect to v. Due to

$$\int dv v^k \frac{\partial}{\partial v}\left(\tilde{\rho}_i \frac{\tilde{V}_{0i}(v) - v}{\tau_i} \right) = -\int dv k v^{k-1} \left(\tilde{\rho}_i \frac{\tilde{V}_{0i}(v) - v}{\tau_i} \right) = -\frac{k\rho_i}{\tau_i}(\langle v^{k-1} v_0 \rangle_i - \langle v^k \rangle_i) \quad (25)$$

and

$$(1-p_i) \int dv \tilde{\rho}_i(r,v,t) \int dw (w v^k - v^{k+1}) \tilde{\rho}_i(r,w,t) = (1-p_i)(\rho_i)^2 (\langle v \rangle_i \langle v^k \rangle_i - \langle v^{k+1} \rangle_i) \quad (26)$$

we obtain the macroscopic moment equations

$$\frac{\partial}{\partial t}(\rho_i \langle v^k \rangle_i) + \frac{\partial}{\partial r}(\rho_i \langle v^{k+1} \rangle_i) = \frac{k\rho_i}{\tau_i}(\langle v^{k-1} v_0 \rangle_i - \langle v^k \rangle_i) + (1-p_i)(\rho_i)^2 (\langle v \rangle_i \langle v^k \rangle_i - \langle v^{k+1} \rangle_i) + \frac{p_{i-1}^+}{T_{i-1}^0} \rho_{i-1} \langle v^k \rangle_{i-1}$$
$$- \frac{p_i^+}{T_i^0} \rho_i \langle v^k \rangle_i + \frac{p_{i+1}^-}{T_{i+1}^0} \rho_{i+1} \langle v^k \rangle_{i+1} - \frac{p_i^-}{T_i^0} \rho_i \langle v^k \rangle_i + \frac{1}{T_{i-1}^+} \rho_{i-1} \langle v^k \rangle_{i-1} - \frac{1}{T_i^+} \rho_i \langle v^k \rangle_i$$
$$+ \frac{1}{T_{i+1}^-} \rho_{i+1} \langle v^k \rangle_{i+1} - \frac{1}{T_i^-} \rho_i \langle v^k \rangle_i + \nu_i^+(r,t) \langle v^k \rangle_i^+ - \nu_i^-(r,t) \langle v^k \rangle_i^- . \quad (27)$$

Here, we have introduced the notation

$$\langle v^k \rangle_i^{\pm} = \int dv \int dv_0 v^k \frac{\hat{\nu}_i^{\pm}(r,v,v_0,t)}{\nu_i^{\pm}(r,t)} = \int dv v^k \frac{\tilde{\nu}_i^{\pm}(r,v,t)}{\nu_i^{\pm}(r,t)} \quad (28)$$

and applied the approximation

$$\int dv\,\widetilde{\rho}_j(r,v,t)\int_{w<v} dw\,v^k(v-w)\widetilde{\rho}_j(r,w,t)\approx\langle v^k\rangle_j\int dv\,\widetilde{\rho}_j(r,v,t)\int_{w<v} dw(v-w)\widetilde{\rho}_j(r,w,t)=\langle v^k\rangle_j\frac{\rho_j(r,t)}{T_j^0}, \quad (29)$$

which is empirically justified due to the smallness of the velocity distributions $P_j(v;r,t)$ (i.e., due to $\sqrt{\theta_j}\ll V_j$ [10,19]).

C. Fluid-dynamic multilane traffic equations

In order to derive dynamic equations for the densities ρ_i and average velocities V_i, we need the relations

$$\begin{aligned}\langle v^2\rangle_i&=\langle[V_i+(v-V_i)]^2\rangle_i\\ &=(V_i)^2+2V_i\langle v-V_i\rangle_i+\langle(v-V_i)^2\rangle_i\\ &=(V_i)^2+\theta_i\end{aligned} \quad (30)$$

and

$$\rho_i\frac{\partial V_i}{\partial t}=\frac{\partial}{\partial t}(\rho_i\langle v\rangle_i)-V_i\frac{\partial\rho_i}{\partial t}. \quad (31)$$

Applying these and and using the abbreviations

$$\frac{1}{\tau_i^{\pm}}=\frac{p_i^{\pm}}{T_i^0}+\frac{1}{T_i^{\pm}}, \quad (32)$$

Eq. (27) gives us the density equations

$$\frac{\partial\rho_i}{\partial t}+V_i\frac{\partial\rho_i}{\partial r}=-\rho_i\frac{\partial V_i}{\partial r}+\nu_i^+(r,t)-\nu_i^-(r,t) \quad (33a)$$

$$+\frac{\rho_{i-1}}{\tau_{i-1}^+}-\frac{\rho_i}{\tau_i^+}+\frac{\rho_{i+1}}{\tau_{i+1}^-}-\frac{\rho_i}{\tau_i^-}. \quad (33b)$$

This result is similar to previous multi-lane models. However, by some lengthy but straightforward calculations we additionally obtain the velocity equations

$$\rho_i\frac{\partial V_i}{\partial t}+\rho_iV_i\frac{\partial V_i}{\partial r}=-\frac{\partial\mathcal{P}_i}{\partial r}+\frac{\rho_i}{\tau_i}(V_i^e-V_i) \quad (34a)$$

$$+\frac{\rho_{i-1}}{\tau_{i-1}^+}(V_{i-1}-V_i)+\frac{\rho_{i+1}}{\tau_{i+1}^-}(V_{i+1}-V_i) \quad (34b)$$

$$+\nu_i^+(V_i^+-V_i)-\nu_i^-(V_i^--V_i) \quad (34c)$$

with the so-called traffic pressures [41,46,47]

$$\mathcal{P}_i=\rho_i\theta_i \quad (35)$$

and the equilibrium velocities

$$V_i^e=V_{0i}-\tau_i(p_i)[1-\rho_i(\rho_i)]\rho_i\theta_i. \quad (36)$$

Equation (34) corrects the phenomenological approach by Michalopoulos *et al.* [28]. The terms containing the rates ν_i^+ and ν_i^- reflect entering and leaving vehicles, respectively. Whereas the terms (33a) and (34a) correspond to the effects of vehicle motion, of acceleration towards the drivers' desired velocities, and of deceleration due to interactions, the terms (33b) and (34b) arise from overtaking and lane-changing maneuvers. Equation (34b) comes from differences between the average velocities in neighboring lanes and tends to reduce them. The term (34c) has a form and interpretation similar to Eq. (34b). It is only negligible if entering vehicles are able to adapt to the velocities in the merging lane and if exiting vehicles initially have an average velocity similar to that in the lane which they are leaving, so that $V_i^{\pm}\approx V_i$.

In order to close Eqs. (33) and (34), we must specify the interaction rates $1/T_i^0$ and the variances θ_i. Utilizing that the empirical velocity distributions $P_i(v;r,t)$ are approximately normally distributed [10,24,44,46], we have

$$P_i(v;r,t)\approx\frac{1}{\sqrt{2\pi\theta_i(r,t)}}e^{-[v-V_i(r,t)]^2/[2\theta_i(r,t)]}, \quad (37)$$

which implies

$$\frac{1}{T_i^0}\approx\rho_i\sqrt{\frac{\theta_i}{\pi}}. \quad (38)$$

With a detailed theoretical and empirical analysis it can be shown [9,19] that the variances $\theta_i(r,t)$ can be well approximated by equilibrium relations $\theta_i^e(\rho_i)$ which are given by the implicit equation

$$\theta_i^e(\rho_i)\equiv\hat{\theta}_{i0}-2\tau_i(\rho_i)[1-p_i^e(\rho_i)]\frac{\rho_i[\theta_i^e(\rho_i)]^{3/2}}{\sqrt{\pi}}. \quad (39)$$

Here, p_i^e denotes the overtaking probability of vehicles on lane i, when the densities ρ_j on the different lanes j are in equilibrium. For the average desired velocities V_{0i} we have

$$V_{0i}\equiv V_{0i}(r,t)\approx\hat{V}_{0i}(r,t) \quad (40)$$

since

$$\hat{P}_{0i}(v_0;r,t)-P_{0i}(v_0;r,t)\approx 0 \quad (41)$$

due to the smallness of T_r.

IV. DERIVATION AND SIMULATION OF A REDUCED MULTILANE MODEL

The velocity equations are mainly needed to model the observed traffic instabilities which lead to the spontaneous formation of stop-and-go waves at medium densities [5–8,10–12]. However, if one is not interested in the density oscillations but only in the *average* temporal evolution of traffic flow, the velocity equations can be eliminated. In order to do this, we will apply a method that has been sug-

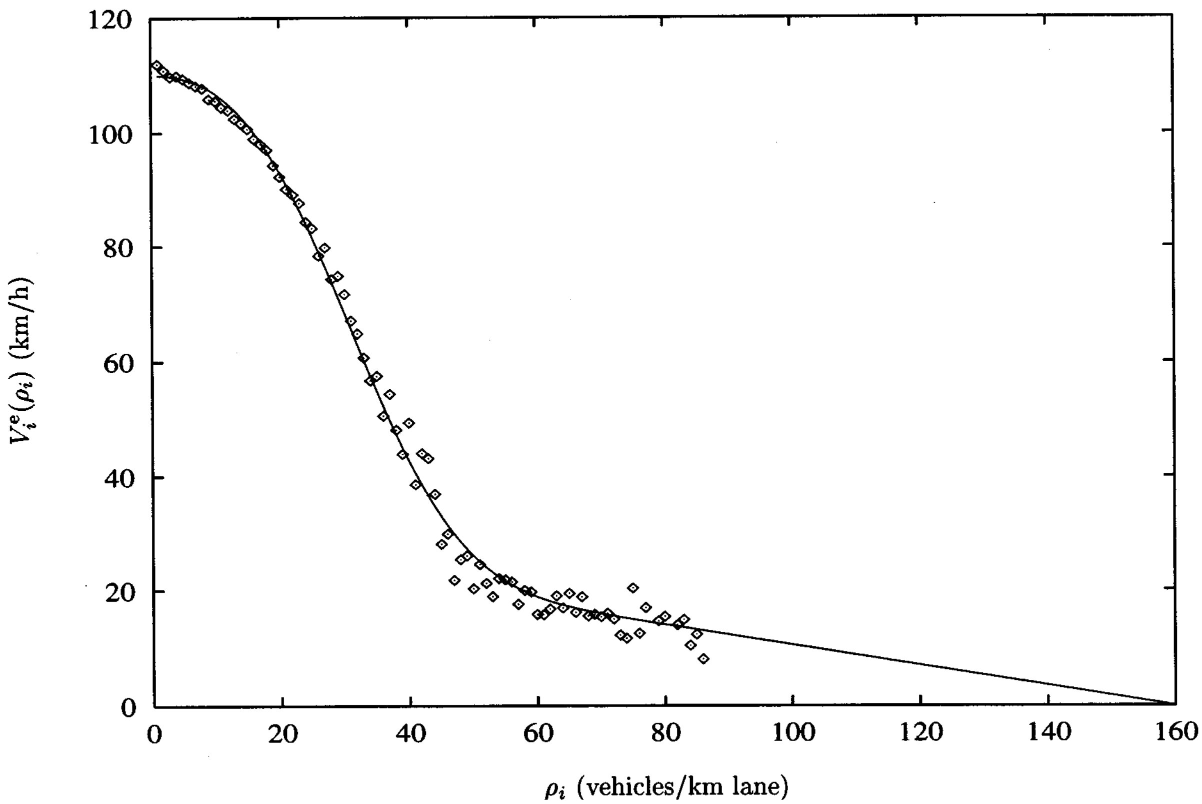

FIG. 1. The chosen velocity-density relation $V_i^e(\rho_i)$ (——) and the corresponding empirical data from the Dutch autobahn A9 ($\diamond$).

gested by Sela and Goldhirsch [48]: First, we introduce the time averages

$$\overline{F_i}(r,t)=\frac{1}{\Delta T}\int_{t-\Delta T/2}^{t+\Delta T/2} dt F_i(r,t) \tag{42}$$

over the least common multiple ΔT of the occurring oscillation periods ΔT_i. Then, the quantities $\overline{\rho_i}(r,t)$ and $\overline{V_i}(r,t)$ will describe the *coarse-grained* traffic dynamics, in other words: the traffic dynamics on a slow time scale. Additionally, the time averages of the total time derivatives $d\rho_i/dt$ and dV_i/dt will approximately vanish:

$$\overline{\frac{d\rho_i}{dt}}\equiv\overline{\frac{\partial\rho_i}{\partial t}}+\overline{V_i\frac{\partial\rho_i}{\partial r}}\approx 0, \quad \overline{\frac{dV_i}{dt}}\equiv\overline{\frac{\partial V_i}{\partial t}}+\overline{V_i\frac{\partial V_i}{\partial r}}\approx 0. \tag{43}$$

This corresponds to the assumption that, in coordinate systems moving with velocities $V_i(r,t)$, the densities $\rho_i(r,t)$ and velocities $V_i(r,t)$ oscillate around their (slowly changing) equilibrium values.

Now, we approximate time averages $\overline{F_i(\rho_i,V_i)}$ of density- and velocity-dependent functions $F_i(\rho_i,V_i)$ by a series in spatial derivatives of $\overline{\rho_i}$ and $\overline{V_i}$. For our purposes it is sufficient to truncate the expansion after the first order [19]:

$$\overline{F_i(\rho_i,V_i)}\approx F_{00}(\overline{\rho_i},\overline{V_i})+F_{10}(\overline{\rho_i},\overline{V_i})\frac{\partial\overline{\rho_i}}{\partial r}+F_{01}(\overline{\rho_i},\overline{V_i})\frac{\partial\overline{V_i}}{\partial r}. \tag{44}$$

With this and Eq. (43) we obtain from the time average of velocity equations (34)

$$\overline{V_i}=\frac{\dfrac{\overline{\rho_i}}{\tau_i}V_i^e(\overline{\rho_i})+\dfrac{\overline{\rho_{i-1}}}{\tau_{i-1}^+}\overline{V_{i-1}}+\dfrac{\overline{\rho_{i+1}}}{\tau_{i+1}^-}\overline{V_{i+1}}-\dfrac{\partial\mathcal{P}_i(\overline{\rho_i})}{\partial\overline{\rho_i}}\dfrac{\partial\overline{\rho_i}}{\partial r}}{\dfrac{\overline{\rho_i}}{\tau_i}+\dfrac{\overline{\rho_{i-1}}}{\tau_{i-1}^+}+\dfrac{\overline{\rho_{i+1}}}{\tau_{i+1}^-}}. \tag{45}$$

Here, we have restricted our considerations to the case of a freeway without entrances and exits. Resolving Eqs. (45) with respect to $\overline{V_i}$ leads to relations of the form

$$\overline{V_i}=\mathcal{V}_i(\{\overline{\rho_j}\})-\sum_k\frac{\mathcal{D}_{ik}(\{\overline{\rho_j}\})}{\overline{\rho_i}}\frac{\partial\overline{\rho_k}}{\partial r}, \tag{46}$$

which only depends on the densities $\overline{\rho_j}$ and their gradients. Inserting this into the density equations (33) finally leads to the reduced equations

$$\frac{\partial\overline{\rho_i}}{\partial t}+\frac{\partial}{\partial r}[\overline{\rho_i}\mathcal{V}_i(\{\overline{\rho_j}\})]=\sum_k\frac{\partial}{\partial r}\left[\mathcal{D}_{ik}(\{\overline{\rho_j}\})\frac{\partial\overline{\rho_k}}{\partial r}\right]+\frac{\overline{\rho_{i-1}}}{\tau_{i-1}^+}-\frac{\overline{\rho_i}}{\tau_i^+}+\frac{\overline{\rho_{i+1}}}{\tau_{i+1}^-}-\frac{\overline{\rho_i}}{\tau_i^-}. \tag{47}$$

If we neglect products of spatial derivatives we end up with the coupled Burgers equations [49]

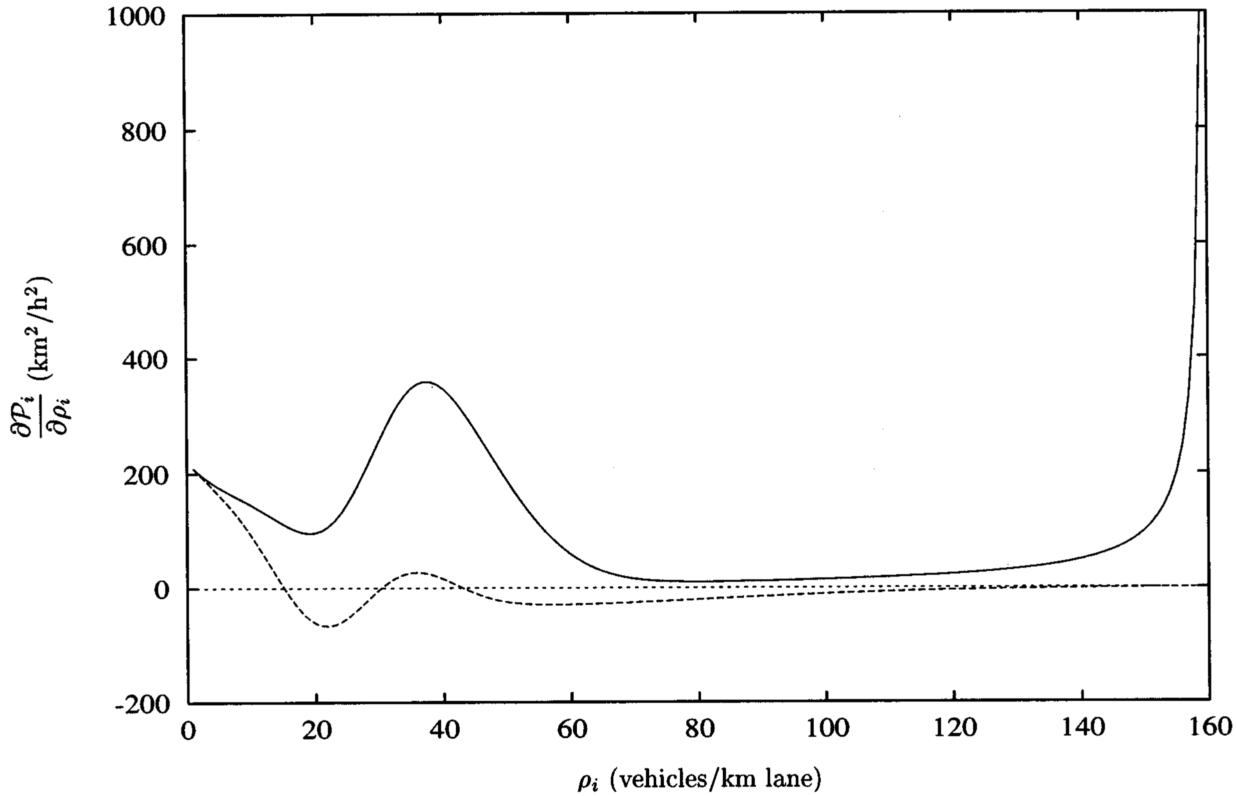

FIG. 2. Representation of the density gradients of the *idealized* traffic pressure $\mathcal{P}_i = \rho \theta_i^e(\rho)$ of point-like vehicles (— —) and the *corrected* pressure relation (——) which takes into account their finite space requirements. Obviously, the increase of the corrected traffic pressure with density and the corrected traffic pressure itself diverge at the maximum density $\rho_{\max}$, so that the latter cannot be exceeded. For this reason, the diffusion functions $\mathcal{D}_{ik}$ also diverge for $\rho_k \to \rho_{\max}$. The pressure relations have been reconstructed from empirical data by means of theoretical relations [10,19].

$$\frac{\partial \overline{\rho_i}}{\partial t} + \frac{\partial}{\partial r}[\overline{\rho_i}\mathcal{V}_i(\{\overline{\rho_j}\})] = \sum_k \mathcal{D}_{ik}(\{\overline{\rho_j}\}) \frac{\partial^2 \overline{\rho_k}}{\partial r^2} + \frac{\overline{\rho_{i-1}}}{\tau_{i-1}^+} - \frac{\overline{\rho_i}}{\tau_i^+} + \frac{\overline{\rho_{i+1}}}{\tau_{i+1}^-} - \frac{\overline{\rho_i}}{\tau_i^-}. \tag{48}$$

On the right-hand side of this equation we have a sum of *diffusion terms* with density-dependent diffusion functions $\mathcal{D}_{ik}$. These cause a smoothing of sudden density changes and prevent the formation of shock waves. This is the reason why density gradients and especially products of spatial derivatives are normally negligible, which justifies the approximation made with Eq. (48). Apart from this, the diffusion terms are very helpful for efficient and stable numerical integration schemes.

We will now focus on the simulation of multilane traffic. As an example, we investigate a two-lane autobahn (i.e., $i \in \{1,2\}$). For reasons of simplicity, in both lanes the velocity-density relations $V_i^e(\rho_i)$ and pressure relations $\mathcal{P}_i(\rho_i)$ will be chosen identically. This is at least justified for congested traffic (with a density of 30 vehicles per kilometer and lane or more). The corresponding relations are depicted in Figs. 1 and 2. They have been reconstructed from empirical data of the Dutch highway A9 between Haarlem and Amsterdam with a speed limit of 120 km/h and take into account corrections of the traffic equations for high densities (for details see Refs. [10,19]).

The lane-changing rates $1/\tau_i^\pm$ are chosen in accordance with an empirically validated model [50]:

$$\frac{1}{\tau_i^\pm} = \beta_i^\pm \overline{\rho_i}(\rho_{\max} - \overline{\rho_{i\pm 1}}). \tag{49}$$

Therefore $1/\tau_i^\pm$ is proportional to the vehicle density $\overline{\rho_i}$ which reflects the grade of obstruction by slower vehicles on lane i. The factor $(\rho_{\max} - \overline{\rho_{i\pm 1}})$ reflects that vehicles can change to the neighboring lane $i \pm 1$ less frequently the more the density on it reaches the *maximum density* $\rho_{\max}$. For German autobahns the parameters $\beta_i^\pm$ have the following values:

$$\beta_1^+ = 0.176 \times 10^{-3}, \quad \beta_2^- = 0.056 \times 10^{-3}, \\ \beta_1^- = \beta_2^+ = 0. \tag{50}$$

The relation $\beta_1^+ > \beta_2^-$ originates from the fact that the left lane is preferred in Germany, since overtaking is forbidden in the right-hand lane.

Bottleneck situations can be simulated in the following way: We will assume that the right lane is closed between places r_0 and r_1. Then, the lane-changing rate β_1^+ will be considerably increased, but β_2^- will be zero on this stretch and already a certain interval Δr before (i.e., for $r_0 - \Delta r \leqslant r \leqslant r_1$). β_1^+ and Δr must be chosen sufficiently large so that the right lane is empty up to the beginning r_0 of

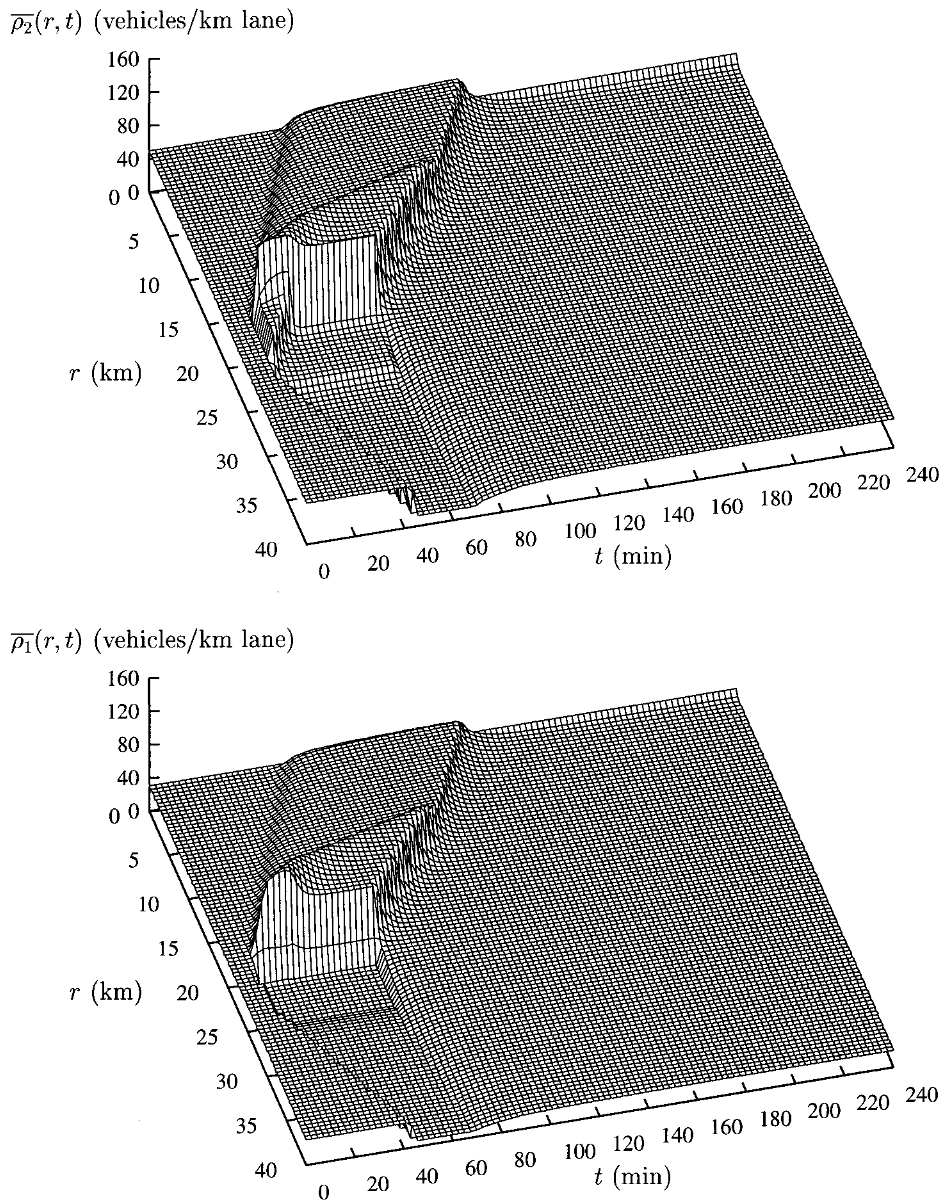

FIG. 3. Spatiotemporal evolution of the time-averaged densities on a two-lane freeway stretch of 40 km length with open boundary conditions in the case of an overloaded temporary bottleneck situation (above: left lane; below: right lane). The right lane is closed between $t=10$ min and $t=60$ min on the stretch between $r_0=21$ km and $r_1=25$ km, and the vehicles in lane 1 try to get into lane 2 beginning at $r_0-\Delta r=20$ km. This suddenly increases the density in the left lane in the region of the bottleneck, whereas the right lane becomes empty. Already after a short time the most extreme clustering develops at the beginning of the bottleneck, where the vehicles of the closed lane try to squeeze in the left lane. Since the capacity of the remaining lane is smaller than the total traffic volume, the left lane becomes overloaded. For this reason a congestion running upstream (tailback) builds up in *both* lanes. On the other hand, the density after the bottleneck, where two lanes are available again, is smaller than in front of it so that the vehicles can accelerate there. As a consequence, the traffic situation already recovers in the course of the bottleneck. At $t=60$ min, the lane closure is lifted and the traffic jam disappears. (Note: Due to the different lane-changing rates the equilibrium density is somewhat greater than 40 vehicles per kilometer and lane in the left lane and somewhat smaller in the right lane.)

the bottleneck. Simulation results for the traffic dynamics above and below capacity are presented in Figs. 3 and 4, respectively.

V. SUMMARY AND OUTLOOK

In this paper we have derived a macroscopic traffic model for unidirectional multilane roads. Our considerations started from plausible assumptions about the behavior of driver-vehicle units regarding acceleration, overtaking, deceleration, and lane-changing maneuvers. The resulting gas-kinetic traffic model is a generalization of Paveri-Fontana's Boltzmann-like traffic equation. It can be extended to situations where different vehicle types or driving styles are to be investigated [19].

The gas-kinetic traffic equations not only allow one to derive dynamic equations for the vehicle density in each lane, but also for the average velocity. In this way we were

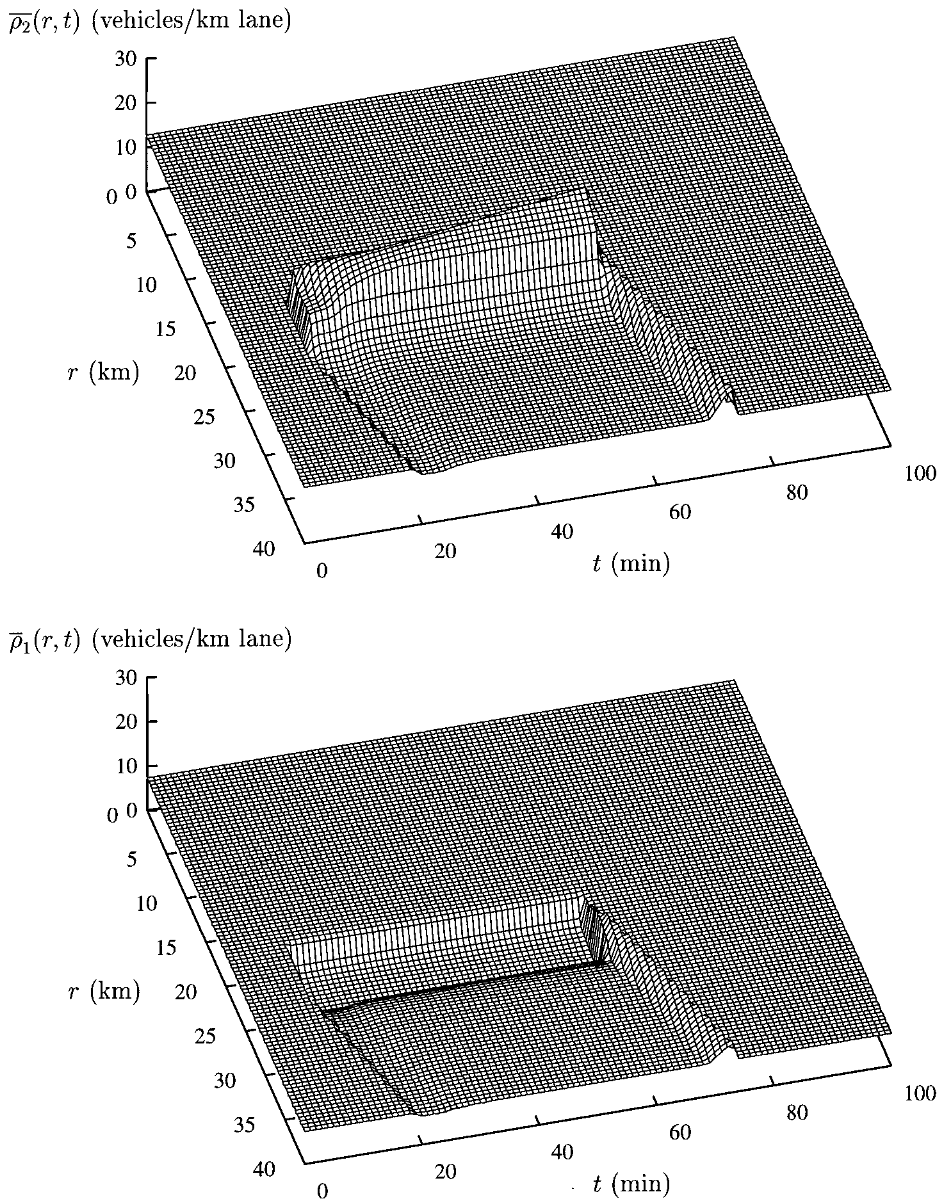

FIG. 4. Same as Fig. 3, but for a bottleneck situation below capacity. At an average density of ten vehicles per kilometer and lane the traffic capacity of one lane is large enough to cope with the total traffic volume. Therefore no congestion running upstream builds up, but in the left lane the density is increased in the region of the bottleneck. After the lane closure is lifted, the traffic jam, which was previously localized at the bottleneck, causes a damped density wave. This propagates along the freeway with a velocity that is slower than the average vehicle velocity. Due to lane-changing maneuvers, the right lane also develops a propagating density wave.

able to extend and correct previous phenomenological multilane models. Overtaking and lane-changing maneuvers are explicitly taken into account, so that the interactions between neighboring lanes are included.

We have then eliminated the velocity equations in order to obtain a reduced model that allows efficient computer simulations. The resulting density equations describe the average temporal evolution of traffic on a slow time scale. They contain diffusion terms which diverge at maximum density $\rho_{\max}$ if the finite space requirements of vehicles are taken into account. This guarantees that $\rho_{\max}$ cannot be exceeded and density shocks are smoothed out. The latter is important for realistic results and stable numerical integration schemes. Finally, the reduced multilane traffic model has been applied to the difficult case of bottleneck situations. The computational results were very plausible. Consequently, the model can be used to investigate a number of questions concerning the optimization of traffic flow.

(1) In what way does on-ramp traffic influence and destabilize the traffic flow in the other lanes? How does the destabilization effect depend on the traffic volume, the length of the on-ramp lane, the total lane number, etc.?

(2) In case of a reduction of the number of lanes, is it better to close the leftmost or the rightmost lane?

(3) Is the organization of American freeways or of European autobahns more efficient, or would a suitable mixture of both be most efficient? Remember that American freeways are characterized by uniform speed limits and the fact that overtaking as well as lane changing are allowed in both neighboring lanes. In contrast, on European autobahns often no speed limit is prescribed (at least in Germany) and average velocity normally increases with growing lane number since overtaking is only allowed in the left-hand lane.

(4) In which traffic situations do stay-in-lane recommendations increase the efficiency of roads?

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